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Dressing transformation method for finding soliton solutions of the sinh–Gordon equation

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Abstract. In this paper, the establishment of the dressing transformations in the sinh–Gordon model is reconsidered. By carefully analysing the infinitesimal structures of dressing transformations, we improve the algebraic method for solving the dressing problem in the system and then lay the dressing transformation method on a firm basis. The modified dressing transformation method, which no longer contains any deductive jumps, turns out to become a powerful Hamiltonian approach to finding N-soliton solutions ($N \geqslant 1$) of the integrable systems.

In recent years the investigations on the group of dressing transformations have excited much interest and attraction in theoretical and mathematical physics. The increase in this research lies mostly in the fact that the dressing action on a phase space is a Poisson–Lie action and a dressing group is the classical precursor of the quantum group structure of an integrable field system in two dimensions [1]. Moreover, this group also plays a fundamental role in our understanding of the classical integrability of nonlinear field systems [2, 3]. The dressing transformations belong to those gauge transformations of an integrable system which act on the Lax connection [4] of the system and preserve its form. They manifestly supply a method for finding the soliton solutions of the integrable nonlinear equations. Pioneering work to develop this dressing transformation method has been undertaken by Babelon and Bernard. The authors succeeded in rediscovering the N-soliton solutions of the sinh–Gordon model by means of the dressing transformations [3].

The basis of the dressing transformation method for finding soliton solutions is to construct the finite forms of dressing transformations. In [3] Babelon and Bernard solved this problem through studying a finite gauge transformation acting on the Lax connection and preserving its form. Because such a gauge transformation is not necessarily a dressing transformation [5–7], Babelon and Bernard's scheme is in a sense questionable. As a matter of fact, the two authors had to introduce some additional assumptions when they implemented their strategy (see section 5.1. of [3]). It is perhaps a worse problem that in practice the Babelon and Bernard's scheme proves not to fulfil the needs of finding the solutions of the Zhiber–Mikhailov–Shabat model [8, 9] and the solutions of some other integrable systems. Obviously, the dressing transformation method should be developed further. In the present paper, we intend to make some attempts in this direction. For convenience of exposition we shall also focus our attention on the sinh–Gordon model. We shall show that the deductive difficulties encountered in [3] can be overcome if we begin by considering the infinitesimal dressing transformations.

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The sinh–Gordon model is a simple and important two-dimensional massive scalar field theory. The classical field equation for this model reads

$$\partial_+\partial_-\phi - 2m^2 \sinh(2\phi) = 0 \tag{1}$$

where m is a mass parameter, $\partial_\pm \equiv \frac{1}{2}(\partial_1 \pm \partial_0)$, and $x_\pm \equiv (x^1 \pm x^0)$ denotes the light cone coordinates in two-dimensional Minkowski space. As is well known, equation (1) can be expressed as the compatibility condition

$$F_{+,-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \tag{2}$$

of the linear systems

$$[\partial_\pm + A_\pm(x, \lambda)]T(x, \lambda) = 0 \tag{3}$$

where $A_\pm(x, \lambda)$ (the components of the Lax connection) are two matrix functionals of the sinh–Gordon field $\phi(x)$,

$$A_\pm(x, \lambda) = \pm\tfrac{1}{2}\partial_\pm\phi H + m\lambda^{\pm 1}(\mathrm{e}^{\pm\phi} E + \mathrm{e}^{\mp\phi} F) \tag{4}$$

(where $H = \sigma_3$, $E = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)$, $F = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)$; σ_1, σ_2 and σ_3 are three Pauli matrices). $T(x, \lambda)$ is the transport matrix of the considered system. Note that a non-trivial *spectral parameter* λ has been remarkably furnished to the Lax connection (4) and that this spectral parameter cannot be removed by a gauge transformation. Therefore, the sinh–Gordon model is a Toda model over the loop algebra $\widetilde{sl}_2$ rather than over the simple Lie algebra sl_2. In order to make the integrability of equation (1) become transparent, we introduce an important ingredient in the Hamiltonian formalism, the so-called classical Lax operator of the sinh–Gordon model:

$$A(x, \lambda) \equiv \tfrac{1}{2}(A_+ + A_-) = \tfrac{1}{4}[\pi_\phi H + 2m\,\mathrm{e}^{\phi}(\lambda E + 1/\lambda F) + 2m\,\mathrm{e}^{-\phi}(\lambda F + 1/\lambda E)]. \tag{5}$$

It goes without saying that the classical Lax operator gives an explicit definition of the transport matrix,

$$T(x, \lambda) \equiv \mathcal{P} \exp\left[- \int_0^{x^1} A(y, \lambda)\, \mathrm{d}y^1 \right].$$

Such a $T(x, \lambda)$ clearly belongs to the loop group $\widetilde{SL}_2$. In (5) the quantity $\pi_\phi(x) \equiv \partial_0\phi(x)$ is the canonical conjugate momentum of the scalar field $\phi(x)$. Therefore, there is a non-trivial Poisson bracket $\{\phi(x), \pi_\phi(y)\} = \delta(x^1 - y^1)$ in phase space. Relying on this bracket, we have

$$\{A(x, \lambda) \overset{\otimes}{,} A(y, \nu)\} = [r(\lambda, \nu), A(x, \lambda) \otimes 1 + 1 \otimes A(y, \nu)]\delta(x^1 - y^1) \tag{6}$$

or equivalently

$$\{T(x, \lambda) \overset{\otimes}{,} T(x, \nu)\} = [r(\lambda, \nu), T(x, \lambda) \otimes T(x, \nu)] \tag{7}$$

where the matrix structure constant

$$r(\lambda, \nu) = \frac{1}{4}\left[\frac{1}{2}\frac{\lambda^2 + \nu^2}{\lambda^2 - \nu^2} H \otimes H + \frac{2\lambda\nu}{\lambda^2 - \nu^2}(E \otimes F + F \otimes E)\right] \tag{8}$$

is a trigonometric solution to the classical Yang–Baxter equation. From these results we see that the fundamental Poisson bracket in the sinh-Gordon model takes an ultralocal form [10] and that $\mathrm{Tr}[T(x, \lambda)]$ generates an infinite number of quantities in involution. These results achieve the Hamiltonian proof of the classical integrability of the sinh–Gordon model.

The complete integrability of a system in two dimensions has close ties with the so-called dressing transformations. The dressing transformations describe some non-canonical symmetries of the system. Establishment of the transformations is related to a factorization problem (the Riemann–Hilbert problem) in the group (underlying the system) specified by the matrices $R_\pm(X) = \mathrm{Tr}_2[r_\pm 1 \otimes X]$ [11]. For the sinh–Gordon model under consideration, in respect to the fact that this model is a Toda model over the loop group $\widetilde{SL}_2$, the infinitesimal Riemann–Hilbert problem should be defined by the following decomposition of the loop algebra $\widetilde{sl}_2$,

$$X_\pm(\lambda) \sim \oint_{\mathcal{C}_\pm} \frac{\mathrm{d}\nu}{2\pi\,\mathrm{i}\nu}\,\mathrm{Tr}_2\left[r_\pm\left(\frac{\lambda}{\nu}\right) 1 \otimes X(\nu)\right]$$

where $X(\lambda)$ is an arbitrary element of $\widetilde{sl}_2$. In the above formula, the integration contour $\mathcal{C}_-$ encircles the singularity $\nu = 0$ while $\mathcal{C}_+$ encircles the singularity $\nu = \infty$. The $r_\pm(\lambda/\nu)$ are projection operators, which respectively correspond to the expansions of the classical $r(\lambda, \nu)$ matrix either in powers of (λ/ν) or in powers of (ν/λ). Note that the loop algebra $\widetilde{sl}_2$ on which the Lax connection (4) takes its value assumes the so-called *principal gradation*,

$$X(\lambda) = \frac{1}{2}\sum_{i=-\infty}^{+\infty} \lambda^{2i}\left(\frac{1}{2}x_{1,i}H + x_{2,i}\lambda E + x_{3,i}\lambda F\right). \tag{9}$$

Let the normalization constant in the definition of the Riemann–Hilbert problem be -2, then

$$X(\lambda) = X_+(\lambda) - X_-(\lambda) \tag{10}$$

and the explicit expressions of $X_\pm(\lambda)$ read

$$X_+(\lambda) = -\frac{1}{8}x_{1,0}H + \frac{1}{2}\sum_{i=0}^{+\infty} \lambda^{2i}\left(\frac{1}{2}x_{1,i}H + x_{2,i}\lambda E + x_{3,i}\lambda F\right)$$

$$X_-(\lambda) = -\frac{1}{8}x_{1,0}H - \frac{1}{2}\sum_{i=-\infty}^{-1} \lambda^{2i}\left(\frac{1}{2}x_{1,i}H + x_{2,i}\lambda E + x_{3,i}\lambda F\right). \tag{11}$$

Such $X_\pm(\lambda)$ are respectively among two different and infinite-dimensional subalgebras of $\widetilde{sl}_2$, and more importantly, the pair $(X_+(\lambda), X_-(\lambda))$ gives an element of the dressing algebra.

For any element $X(\lambda) \in \widetilde{sl}_2$ with factorization given by equations (9) and (11), the infinitesimal dressing transformations of equation (1) could be elaborately designed as a special kind of gauge transformation acting on the Lax connection (4),

$$\delta A_\alpha(x, \lambda) = [\Theta_\pm(x, \lambda), A_\alpha(x, \lambda)] - \partial_\alpha \Theta_\pm(x, \lambda) \qquad (\alpha = \pm) \tag{12}$$

where $\Theta_\pm(x, \lambda)$ are defined by the factorization of the element $T(x, \lambda)X(\lambda)T^{-1}(x, \lambda)$

$$T(x, \lambda)X(\lambda)T^{-1}(x, \lambda) \equiv \Theta(x, \lambda) = \Theta_+(x, \lambda) - \Theta_-(x, \lambda) \tag{13}$$

in $\widetilde{sl}_2$. $T(x, \lambda)$ is of course the aforesaid transport matrix. Relying on the zero-curvature equation (2), the gauge transformations governed by (12) are indeed the symmetric transformations of equation (1). Suppose that

$$\Theta(x, \lambda) = \frac{1}{2}\sum_{i=-\infty}^{+\infty} \lambda^{2i}\left[\frac{1}{2}\theta_{1,i}(x)H + \theta_{2,i}(x)\lambda E + \theta_{3,i}(x)\lambda F\right] \tag{14}$$

then

$$\Theta_+(x, \lambda) = -\frac{1}{8}\theta_{1,0}H + \frac{1}{2}\sum_{i=0}^{+\infty} \lambda^{2i}\left(\frac{1}{2}\theta_{1,i}H + \theta_{2,i}\lambda E + \theta_{3,i}\lambda F\right)$$

$$\Theta_-(x, \lambda) = -\frac{1}{8}\theta_{1,0}H - \frac{1}{2}\sum_{i=-\infty}^{-1} \lambda^{2i}\left(\frac{1}{2}\theta_{1,i}H + \theta_{2,i}\lambda E + \theta_{3,i}\lambda F\right). \tag{15}$$

The transform parameters $\theta_{\alpha,i}(x)(\alpha = 1, 2, 3; i = 0, \pm 1, \pm 2, \ldots, \pm\infty)$ are some non-local functionals of the canonical variables (in phase space). Among these transform parameters, $\theta_{1,0}(x)$ is shown from equations (3), (4) and (12), to be the very infinitesimal variation of the sinh–Gordon field $\phi(x)$,

$$\theta_{1,0}(x) = -4\delta_X\phi(x) \approx -4[\widetilde{\phi}(x) - \phi(x)]. \tag{16}$$

The other parameters are governed by two systems of differential equations:

$$\begin{cases} \partial_+\theta_{1,i} - 2m\theta_{2,i-1}\,\mathrm{e}^{-\phi} + 2m\theta_{3,i-1}\,\mathrm{e}^{\phi} = 0 \\ \partial_+\theta_{2,i} + \theta_{2,i}\partial_+\phi - m\theta_{1,i}\,\mathrm{e}^{\phi} = 0 \\ \partial_+\theta_{3,i} - \theta_{3,i}\partial_+\phi + m\theta_{1,i}\,\mathrm{e}^{-\phi} = 0 \end{cases} \qquad \begin{cases} \partial_-\theta_{1,i} - 2m\theta_{2,i}\,\mathrm{e}^{\phi} + 2m\theta_{3,i}\,\mathrm{e}^{-\phi} = 0 \\ \partial_-\theta_{2,i} - \theta_{2,i}\partial_-\phi - m\theta_{1,i+1}\,\mathrm{e}^{-\phi} = 0 \\ \partial_-\theta_{3,i} + \theta_{3,i}\partial_-\phi + m\theta_{1,i+1}\,\mathrm{e}^{\phi} = 0. \end{cases} \tag{17}$$

The determination of the infinitesimal dressing problem in sinh–Gordon theory would just reduce to an exercise to find the solutions of equations (17). In other words, to establish the infinitesimal dressing transformations for the sinh–Gordon model means to express all the parameters $\theta_{\alpha,i}$ of the matrices (15) with the infinitesimal variation $\delta_X\ \phi(x)$ of the field $\phi(x)$.

It is an instructive exercise to recast equation (16), as well as its time derivative, as follows:

$$\begin{aligned} \delta_X\phi(x) &= \frac{1}{4}\theta_{1,0}(x) = -\frac{1}{\pi\mathrm{i}}\oint_{\mathcal{C}_-}\frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{Tr}[X(\lambda)T^{-1}(y,\lambda)\{\phi(x), T(y,\lambda)\}] \\ \delta_X\pi_\phi(x) &= \frac{1}{4}\partial_0\theta_{1,0}(x) = -\frac{1}{\pi\mathrm{i}}\oint_{\mathcal{C}_-}\frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{Tr}[X(\lambda)T^{-1}(y,\lambda)\{\pi_\phi(x), T(y,\lambda)\}] \qquad (x^1 < y^1). \end{aligned} \tag{18}$$

These two formulae indicate that the variations of the canonical conjugate pair under dressing transformations are generated by the transport matrix $T(x,\lambda)$ (in a nonlinear way). Such generation actions are clearly not canonical actions. As a matter of fact, the actions on the phase space of the dressing transformations are Poisson–Lie actions instead. They do not preserve the original symplectic form of phase space or (equivalently) the fundamental Poisson brackets, unless an additional Poisson structure is introduced in the group of transformations. Let $\widetilde{\phi}(x)$ and $\widetilde{\pi}_\phi(x)$ be the dressed field variables. The fundamental Poisson brackets transform covariantly only if $\{\widetilde{\phi}(x), \widetilde{\pi}_\phi(y)\} = \delta(x^1 - y^1)$. Infinitesimally, we have $\widetilde{\phi}(x) = \phi(x) + \delta_X\phi(x)$ and $\widetilde{\pi}_\phi(x) = \pi_\phi(x) + \delta_X\pi_\phi(x)$. The covariance condition becomes

$$\{\phi(x), \delta_X\pi_\phi(y)\} + \{\delta_X\phi(x), \pi_\phi(y)\} + \{\delta_X\phi(x), \delta_X\pi_\phi(y)\}_{G_R} = 0 \tag{19}$$

where $\{\,,\,\}_{G_R}$ is the Poisson bracket furnished on a dressing group. Substituting (18) into (19) gives

$$\{X(\lambda) \overset{\otimes}{,} X(\nu)\}_{G_R} = [\mathcal{C}(\lambda/\nu), X(\lambda)\otimes 1] \tag{20}$$

where $\mathcal{C}(\lambda/\nu)$ is a Casimir-like operator of the loop algebra $\widetilde{sl}_2$,

$$\mathcal{C}(\lambda/\nu) = -\frac{1}{4}\sum_{-\infty}^{+\infty}\left(\frac{\lambda}{\nu}\right)^{2i}\left[H\otimes H + 2\left(\frac{\lambda}{\nu}\right)(E\otimes F + F\otimes E)\right]. \tag{21}$$

It is not difficult to check that such a Poisson bracket does satisfy the antisymmetry and Jacobian identity. By employing the expression (10) of $X(\lambda)$, (20) can be expanded as

$$\begin{aligned} &\{x_{1,i}, x_{1,j}\}_{G_R} = 0 \qquad \{x_{1,i}, x_{2,j}\}_{G_R} = 2x_{2,i+j} \qquad \{x_{1,i}, x_{3,j}\}_{G_R} = -2x_{3,i+j} \\ &\{x_{2,i}, x_{2,j}\}_{G_R} = 0 \qquad \{x_{2,i}, x_{3,j}\}_{G_R} = x_{1,i+j+1} \qquad \{x_{3,i}, x_{3,j}\}_{G_R} = 0 \end{aligned}$$

which is clearly the classical precursor of the *quantum algebra* $U_q(\widetilde{sl}_2)$. The transform parameters $x_{\alpha,i}$ ($\alpha = 1, 2, 3; i = 0, \pm1, \pm2, \ldots, \pm\infty$) will become the operators acting on an infinite-dimensional *quantum Fock space* after quantization. By comparison, the generators $\lambda^{2i}H, \lambda^{2i+1}E, \lambda^{2i+1}F (i = 0, \pm1, \pm2, \ldots, \pm\infty)$ of the loop algebra $\widetilde{sl}_2$ are merely the operators acting on an infinite-dimensional auxiliary vector space [12].

The transform parameters $\theta_{\alpha,i} (\alpha = 1, 2, 3; i = 0, \pm1, \pm2, \ldots, \pm\infty)$ form an infinite sequence of non-local functionals of the canonical variables (of the sinh–Gordon model). They are generally independent of each other unless they are on the solution manifold of equation (1). On the solution manifold (x_+, x_-) there exists a close relationship among these parameters. Let us focus on the infinitesimal dressing transformations of the vacuum solution ($\phi_{\text{vac}}(x) = 0$). In this case, equations (17) reduce to

$$\begin{cases} \partial_+\theta_{1,i} = 2m\theta_{2,i-1} - 2m\theta_{3,i-1} \\ \partial_+\theta_{2,i} = m\theta_{1,i} \\ \partial_+\theta_{3,i} = -m\theta_{1,i} \end{cases} \qquad \begin{cases} \partial_-\theta_{1,i} = 2m\theta_{2,i} - 2m\theta_{3,i} \\ \partial_-\theta_{2,i} = m\theta_{1,i+1} \\ \partial_-\theta_{3,i} = -m\theta_{1,i+1} \end{cases} \tag{22}$$

with $\theta_{1,0}(x) = 4\delta_X\phi(x)$ and $(i, j = 0, \pm1, \pm2, \ldots, \pm\infty)$. On the other hand, it follows directly from equation (1) that the infinitesimal variation $\delta_X\phi(x)$ of the sinh–Gordon vacuum approximately obeys a linear Klein–Gordon equation,

$$\partial_+\partial_-\delta_X\phi = 4m^2\delta_X\phi$$

or equivalently, the following systems of first-order equations

$$\partial_+\delta_X\phi = 2m\mu\delta_X\phi \qquad \partial_-\delta_X\phi = \frac{2m}{\mu}\delta_X\phi \tag{23}$$

(μ is an arbitrary non-vanishing constant). Combining equations (22) with (23) leads to

$$\theta_{1,i} = -4\mu^{-2i}\delta_X\phi(x) \qquad \theta_{2,i} = -\theta_{3,i} = -2\mu^{-(2i+1)}\delta_X\phi(x)$$
$$(i = 0, \pm1, \pm2, \ldots, \pm\infty). \tag{24}$$

Therefore, the infinitesimal dressing transformation of the sinh–Gordon vacuum is either $\Theta^+_\mu(x, \lambda) = -\delta_X\phi(x)V^{(+)}_\mu$ or $\Theta^-_\mu(x, \lambda) = \delta_X\phi(x)V^{(-)}_\mu$, where

$$V^{(\pm)}_\mu = \frac{1}{2}H + \frac{(\lambda/\mu)^{\pm2}}{1 - (\lambda/\mu)^{\pm2}}H + \frac{(\lambda/\mu)^{\pm1}}{1 - (\lambda/\mu)^{\pm2}}(E - F). \tag{25}$$

These $V^{(\pm)}_\mu$ are in full accord with that obtained by Babelon and Bernard [3]. Nevertheless, in the present demonstration no deductive jumps exist. The dressing transformation method for finding the soliton solutions of equation (1) is now laid on a firm basis.

Having formulae (25), the finite forms of the dressing transformations which connect the *single*-soliton solution of equation (1) to vacuum can be simply established by virtue of the *exponential mapping* technique of classical group theory. In addition, it has been proved that the N-soliton solutions of equation (1) could also be extracted from its vacuum solution by such transformations [3]. The finite forms of the dressing transformations acting on the vacuum Lax connection read

$$A_\alpha(0, \lambda) \longrightarrow A^g_\alpha(\phi, \lambda) = g_\pm(x)A_\alpha(0, \lambda)g^{-1}_\pm(x) - \partial_\alpha g_\pm(x)g^{-1}_\pm(x) \qquad (\alpha = \pm) \tag{26}$$

where

$$g^{-1}_-(x)g_+(x) \equiv T_{\text{vac}}(x, \lambda)g^{-1}_-(0)g_+(0)T^{-1}_{\text{vac}}(x, \lambda). \tag{27}$$

$T_{\text{vac}}(x, \lambda)$ is the transport matrix for the vacuum solution of equation (1). The element of the dressing group which maps the vacuum solution into a non-trivial solution of equation (1) is then identified to be $(g_-, g_+) = (g_-(0), g_+(0))$. From equations (3) and (4) we see that the vacuum transport matrix $T_{\mathrm{vac}}(x, \lambda)$ is a local function of the spacetime,

$$T_{\mathrm{vac}}(x, \lambda) = \exp(-mx_+\varepsilon_+)\exp(-mx_-\varepsilon_-). \tag{28}$$

Therefore, the dependence in spacetime of $g_-^{-1}(x)g_+(x)$ is dressed by commuting the factor $\mathrm{e}^{-mx_-\varepsilon_-}\,\mathrm{e}^{-mx_+\varepsilon_+}$ through $g_-^{-1}(0)g_+(0)$,

$$g_-^{-1}(x)g_+(x) \equiv \mathrm{e}^{-mx_-\varepsilon_-}\,\mathrm{e}^{-mx_+\varepsilon_+}g_-^{-1}(0)g_+(0)\,\mathrm{e}^{mx_+\varepsilon_+}\,\mathrm{e}^{mx_-\varepsilon_-}. \tag{29}$$

In view of the infinitesimal dressing transformations of the vacuum solution, the simplest finite dressing transformations under consideration might be assumed to take the forms

$$g_-^{-1}(x) = \exp[\widetilde{\phi}(x)V_\mu^{(-)}] \qquad g_+(x) = \exp[\widetilde{\phi}(x)V_\mu^{(+)}] \tag{30}$$

where $V_\mu^{(\pm)}$ are two special elements of loop algebra $\widetilde{sl}_2$ involved in the considered dressing problem, which have been defined in (25). The unknown function $\widetilde{\phi}(x)$ is a functional of the sinh–Gordon field in general. Owing to an elegant technique developed by Babelon and Bernard [3], we find that the function is nothing but the single-soliton solution of (1),

$$\widetilde{\phi}(x) = 4\arctan\left\{\exp\left[2m\left(\mu x_+ + \frac{x_-}{\mu} + C\right)\right]\right\}. \tag{31}$$

The matrices (30) (with (31)) which give an element $g_\pm = g_\pm(0)$ of the dressing group (in the orbit of the sinh–Gordon vacuum) turn out to be the dressing transformations connecting the vacuum solution of equation (1) to its one-soliton solution. These are the results expected. Following the strategy proposed by Babelon and Bernard [3], these two matrices can be alternately expressed as

$$g_-^{-1}(x) = \exp[\widetilde{\phi}(x)V_\mu^{(-)}]\exp[\tfrac{1}{2}\widetilde{\rho}(x)H] \qquad g_+(x) = \exp[-\tfrac{1}{2}\widetilde{\rho}(x)H]\exp[\widetilde{\phi}(x)V_\mu^{(+)}] \tag{32}$$

where $\widetilde{\rho}(x)$ is another *single*-soliton solution of (1). $\widetilde{\rho}(x)$ is related to $\widetilde{\phi}(x)$ through the following Bäcklund transformation,

$$\begin{aligned} \partial_+(\widetilde{\phi} - \widetilde{\rho}) &= 2m\mu' \sinh(\widetilde{\phi} + \widetilde{\rho}) \\ \partial_-(\widetilde{\phi} + \widetilde{\rho}) &= \frac{2m}{\mu'} \sinh(\widetilde{\phi} - \widetilde{\rho}). \end{aligned} \tag{33}$$

Note that a new transform parameter $\mu'(\mu' \neq \mu)$ has been introduced in (33). Making use of these two formulae, one could achieve the conclusion that the N-soliton solutions of the sinh–Gordon equation are also in the orbit of the vacuum (under the dressing group). Just as Babelon and Bernard did in [3], the dressing transformations which transform the vacuum of equation (1) to its N-soliton solution can be elaborated as the products of N factors,

$$\begin{aligned} g_-^{-1}(N, x) &= g_{1,-}^{-1}(x)g_{2,-}^{-1}(x)\ldots g_{N,-}^{-1}(x) \\ g_+(N, x) &= g_{N,+}(x)g_{N-1,+}(x)\ldots g_{1,+}(x) \end{aligned} \tag{34}$$

where each factor is given by

$$\begin{aligned} g_{k,-}^{-1}(x) &= \exp[(\varphi_k - \varphi_{k-1})V_{\mu_k}^{(-)}]\exp[\tfrac{1}{2}(\varrho_k - \varrho_{k-1})H] \\ g_{k,+}(x) &= \exp[-\tfrac{1}{2}(\varrho_k - \varrho_{k-1})H]\exp[(\varphi_k - \varphi_{k-1})V_{\mu_k}^{(+)}]. \end{aligned} \tag{35}$$

The element $(g_-(k), g_+(k))$ of the dressing group which connects the k-soliton solution of (1) to its vacuum solution is simply given by the elements $g_\pm(k, 0)$ of the loop group $\widetilde{SL}_2$. In (35), the functions $\varphi_k(x)$ and $\varrho_k(x)$ $(k = 0, 1, 2, \ldots, N)$ are just the k-soliton solutions of (1) (with vacuum solution $\varphi_0 \equiv \varrho_0 \equiv 0$). Repeatedly applying the Babelon–Bernard technique to (35) leads to a series of ingenious Bäcklund transformations,

$$\begin{aligned} &\partial_+(\varphi_k - \varrho_k) = 2m\mu_{k+1}\sinh(\varphi_k + \varrho_k) \\ &\partial_-(\varphi_k + \varrho_k) = \frac{2m}{\mu_{k+1}}\sinh(\varphi_k - \varrho_k) \\ &\partial_+(\varphi_k - \varrho_{k-1}) = 2m\mu_k\sinh(\varphi_k + \varrho_{k-1}) \\ &\partial_-(\varphi_k + \varrho_{k-1}) = \frac{2m}{\mu_k}\sinh(\varphi_k - \varrho_{k-1}) \end{aligned} \tag{36}$$

which enables us to find these soliton solutions reiteratively.

In conclusion, we have made a re-investigation into the finite dressing symmetry of the sinh–Gordon model. Such an investigation should not be regarded as a simple copy of [3]. The most evident difference between the two approaches lies in the establishment of the elements $V_\mu^{(\pm)}$ of the loop algebra $\widetilde{sl}_2$, which is a key ingredient of the dressing transformation method in the sinh–Gordon model. In the extractions of $V_\mu^{(\pm)}$, we begin by considering the infinitesimal dressing transformations, whereas Babelon and Bernard started with the finite gauge transformations which act on the Lax connection (4) and preserve its form. As we have indicated, a gauge transformation acting on a Lax connection and preserving its form is not necessarily a dressing transformation. There appears to be some deductive obstacles in [3]. In a sense, our work is an improvement of the Babelon and Bernard's approach. This modified dressing transformation method is an independent Hamiltonian approach to the theory of solitons which provides an algebraic version of the Zakharov–Shabat scheme [2]. It is obviously applicable to finding the soliton solutions of all the ultralocal systems with integrability [10].

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